Diffusional mechanism of strong selection in Ostwald ripening

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The purpose of this paper is to show through a systematic asymptotic analysis that fluctuations, accounted for as a diffusional perturbation in the Lifshitz-Slyozov-Wagner (LSW) model of Ostwald ripening, provides, as conjectured previously by Meerson [Phys. Rev. E **60**, 3072 (1999)], a ''strong'' selection of the limiting solution, out of a one-parameter family of similarity solutions with a finite support, as the sole attractor of time evolution. Throughout the latter, the previously described weak selection of other similarity solutions of that family, by the initial conditions with finite supports, occurs as intermediate time asymptotics. The respective mechanism is traced first for a simple instance of the LSW model with linear characteristic equations (integer power in the particle growth rate law equals -1), beginning with the analysis of steady states in the perturbed problem in similarity variables and weak selection in the unperturbed problem, followed by a detailed asymptotic analysis of the time-dependent perturbed problem, and generalized next for an arbitrary integer power in the range $[-1,2]$. The approximate asymptotic solutions obtained are compared with the exact numerical ones.

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I. INTRODUCTION

Ostwald ripening (OR) is a term pertaining to the coarsening of the particles of the condensed phase in a two phase mixture which forms at the late stage of a first order phase transition. During this stage, due to decrease of the equilibrium substrate concentration (partial vapor pressure) with the increase of particle radius, particles larger than some critical size grow at the expense of the smaller ones. The mean field approach to this process, initiated by Todes [1], has been finalized independently by Lifshitz and Slyozov and Wagner [2], [3] in their respective models of OR. These models consist of continuity equation, for the particle size distribution function $f(r,t)$ in the size space, of the general form

$$f_t+(v(r,t)f)_r=0, \quad 0<r<\infty, \tag{1}$$

where $v(r,t)$ is the growth rate of a particle of radius r specified below, and an equation of the total substrate mass conservation in both phases (solution and the condensed phase). Through the assumption that majority of the substrate is in the condensed phase, valid at the late stage of solution decomposition, this latter equation is further reduced to that of the total particle mass (volume) conservation of the form (in scaled variables):

$$\int_0^\infty r^3 f(r,t)\,dr=1. \tag{2}$$

Finally, particle growth rate $v(r,t)$ in Eq. (1) is usually specified, through some sort of quasi-steady-state reasoning, in the form

$$v(r,t)=\frac{1}{r^N}\left(\frac{1}{R(t)}-\frac{1}{r}\right). \tag{3}$$

Here, N is some integer power in the range $2\geqslant N\geqslant -1$ (see Refs. [4] and [5]) which depends on the substrate transport mechanism, and $R(t)$ is the unknown time-dependent critical particle radius, to be determined as a part of solution together with $f(r,t)$, evolving from a given initial particle size distribution

$$t=0: \quad f(r,0)=f_0(r). \tag{4}$$

Equations (1)–(3) admit a similarity transformation $(r,t,R,f)\leftrightarrow(x,\tau,\alpha,u)$ where

$$\tau=\ln t, \tag{5a}$$

$$x=rt^{-1/(N+2)}, \tag{5b}$$

$$R=t^{1/(N+2)}/\alpha, \tag{5c}$$

$$u(x,\tau)=t^{4/(N+2)}f(r,t). \tag{5d}$$

In terms of similarity variables Eqs. (1)–(3) assume the form

$$u_\tau-\frac{x}{N+2}u_x-\frac{4}{N+2}u+\left(\frac{1}{x^N}\left[\alpha(\tau)-\frac{1}{x}\right]u\right)_x=0 \tag{6a}$$

$$\int_0^\infty ux^3\,dx=\text{const}. \tag{6b}$$

Equations (6) possess a one-parameter family of τ-independent similarity solutions [4,5] for every $N\geqslant -1$, parametrized by the constant critical size parameter α varying in a certain range. Thus, for $N=1$ (Lifshitz-Slyozov model of OR) we have

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$$\left(\frac{3}{2}\right)^{2/3} \leqslant \alpha < 2\left(\frac{3}{5}\right)^{2/3}, \tag{7}$$

$$u = Cx^2 \exp\left(\int_0^x \frac{-6s^2+3\alpha}{s^3-3\alpha s+s} ds \right) \tag{8}$$

with constant C defined by condition (6b). For $(\frac{3}{2})^{2/3} < \alpha < 2(\frac{3}{5})^{2/3}$, Eq. (8) yields

$$u = \begin{cases} Cx^2(\alpha_2 - x)^{\gamma_0}(x-\alpha_1)^{\gamma_2-\gamma_1}(\alpha_3 - x)^{-\gamma_1-\gamma_2}, & \text{for } 0 < x < \alpha_2, \\ 0, & \text{for } x \geqslant \alpha_2, \end{cases} \tag{9a}$$

$$\alpha_1 < 0, \quad 0 < \alpha_2 < \alpha_3. \tag{9b}$$

Here, $\gamma_0 = (5v_0 - 6)/(3 - 2v_0)$, $\gamma_1 = (12 - 7v_0)/(6 - 4v_0)$, $\gamma_2 = 3v_0/[(6-4v_0)s]$, $v_0 = \alpha\alpha_2$, $s = \sqrt{3/(\alpha\alpha_2)}$ and α_1, α_2, α_3 are the respective roots of the equation $x^3 - 3\alpha x + 3 = 0$.

In their original paper [2], Lifshitz and Slyozov argued through heuristic arguments that only one limiting solution of this family, corresponding to

$$\alpha = \left(\frac{3}{2}\right)^{2/3}, \quad \alpha_2 = \alpha_3 = \left(\frac{3}{2}\right)^{1/3}, \quad \alpha_1 = -2\alpha_2, \tag{10a}$$

$$u = \begin{cases} u = Cx^2\left(\frac{\alpha_2 - x}{x+2\alpha_2}\right)^{7/3} \exp\left(-\frac{\alpha_2}{\alpha_2 - x}\right) & \text{for } 0 \leqslant x < \alpha_2, \\ 0 & \text{for } x \geqslant \alpha_2 \end{cases} \tag{10b}$$

is the attractor in time for the system (6a), (6b) for any admissible set of initial states compatible with Eq. (6b). Subsequent studies showed that this was not the case, and any of the finite support solutions with α in the range $(\frac{3}{2})^{2/3} < \alpha < 2(\frac{3}{5})^{2/3}$ are in fact stable for suitable perturbations (e.g., with a support smaller then that for the basic solution). Meerson *et al.* [5] suggested by formal heuristic arguments the following ''weak'' selection criterium of these solutions: the power with which a time dependent solution profile approaches zero at the support boundary is preserved throughout the time evolution and, thus, determines the asymptotic value of α. In other words, the similarity solution the system asymptotically evolves to in time is picked by the respective power in the initial condition with a finite support. This principle was rigorously proved by Carr and Penrose [6] for the case $N = -1$, possibly relevant for sintering in a Stokes flow [6] and dynamics of certain branched polymer systems [7].

Ambiguity of the weak selection, resulting from the nonuniqueness of time asymptotics, dependant on such an essentially unmeasurable parameter as the power in the initial profile at the support boundary, is likely removed by the strong selection principle, also conjectured by Meerson on the intuitive ground. The essence of this principle is as follows.

It has been commonly accepted (since the works of Zeldovich [8] and Frenkel [9]) that the account of fluctuation in the system under consideration amounts to a perturbation of the Eq. (1) by a diffusion term. Meerson conjectured [10] that namely this perturbation provides the ''strong'' selection of the limiting Lifshitz-Slyozov-Wagner (LSW) similarity solution, corresponding to the extreme left value of parameter α, as the sole asymptotic attractor in the problem (1) with a finite initial support and a power low approach of its boundary by the initial profile.

It is our purpose in this paper to demonstrate through a systematic asymptotic analysis that this strong selection principle indeed holds and trace the detailed mechanism by which the critical size parameter evolves from its arbitrary initial value, through that corresponding to a finite support in the unperturbed problem, to its final limiting LSW value. We choose, for the sake of simplicity, to introduce the diffusional perturbation directly into similarity formulation (6) (as is observed in due course, this neither restricts generality nor changes qualitatively the results). Thus, the perturbed problem reads

$$u_\tau - \frac{x}{N+2}u_x - \frac{4}{N+2}u + \left(\frac{1}{x^n}\left[\alpha(\tau) - \frac{1}{x}\right]u\right)_x = \varepsilon^2 u_{xx}, \tag{11a}$$

$$\int_0^\infty ux^3 dx = \text{const}. \tag{11b}$$

We begin with a detailed study of the case $N = -1$. Analysis of this simplest case allows us to infer the universal mechanism of the critical radius evolution in a diffusionally perturbed problem, yielding an approximate asymptotic solution which is compared with the exact numerical one. The very same evolution mechanism is then shown to be valid for the general case $N \geqslant -1$, with the respective approximate and exact numerical solutions compared for the LS case ($N = 1$).

Our presentation has the following structure. In Sec. II we carry out a detailed solution of the unperturbed problem for $N = -1$ and study the $\varepsilon \to 0$ limit of the steady state solutions to the respective perturbed problem for Eqs. (11a), (11b). It is shown that the limiting LSW solution, as opposed to its similarity counterparts with a finite support, belongs to the set of limiting steady-state solutions to the perturbed problem. [In addition to the LSW solution $u = 81\exp(-3x)$ with exponential decay at infinity, this set also includes a family of solution with algebraic decay, particular for the case $N = -1$.] Furthermore, in Sec. III we describe the mechanism for time evolution of the critical size parameter in the diffusionally perturbed problem for $N=-1$, and, based on this, develop an approximate asymptotic solution to the time dependent perturbed problem, exhibiting the strong selection property. This solution is compared to the exact numerical one.

II. WEAK SELECTION IN THE OSTWALD-RIPENING MODEL WITH $N=-1$ AND STEADY-STATE SOLUTIONS IN THE DIFFUSIONALLY PERTURBED VERSION OF THIS MODEL

In this section we present a detailed study of the unperturbed equation (6a) with $N=-1$ and of the steady-state solutions for the respective diffusionally perturbed equation (11a). Particularity of the case $N=-1$ stems from the fact that the characteristic equations for Eq. (6a) are linear in this case and the solution may be found analytically.

The respective model problem reads

$$u_\tau-3u+\{[x(\alpha(\tau)-1)-1]u\}_x=0, \quad 0<x<\infty, \tag{12a}$$

$$u|_{\tau=0}=u_0(x), \tag{12b}$$

$$\int_0^\infty x^3 u\,dx=\text{const.} \tag{12c}$$

This problem has been studied by Carr and Penrose [6] for a simplified conservation condition (12c) of the form

$$\int_0^\infty xu\,dx=\text{const.} \tag{12d}$$

For completeness, we rederive below the respective solution of the problem (12a), (12b) with the original conservation condition (12c).

Let us recall first that the one-parameter family of the steady-state solutions to this problem, parametrized by the critical size parameter $\alpha>0$, is

$$1<\alpha<4,\ u=\begin{cases} C[1-x(\alpha-1)]^{(4-\alpha)/(\alpha-1)} & \text{for } 0\leqslant x<\dfrac{1}{\alpha-1}\\ 0 & \text{for } x\geqslant\dfrac{1}{\alpha-1},\end{cases}$$

$$\alpha=1,\quad u=C\exp(-3x)\quad \text{for } x\geqslant 0,$$

$$0<\alpha<1,\quad u=C([1-\alpha]x+1)^{-(4-\alpha)/(1-\alpha)}\ \text{for } x\geqslant 0. \tag{13}$$

Furthermore, let us simplify the problem (12a)–(12c) by integrating Eq. (12a) four times, while defining a new unknown:

$$Q(x,\tau)\overset{def}{=}\int_x^\infty\int_s^\infty\int_p^\infty\int_q^\infty u(l,\tau)\,dl\,dq\,dp\,ds. \tag{14}$$

Equation (12a) is rewritten accordingly as

$$Q_\tau-3\alpha(\tau)Q+\{x[\alpha(\tau)-1]-1\}Q_x=0, \quad 0<x<\infty, \tag{15a}$$

with the initial condition

$$Q|_{\tau=0}=Q_0(x) \tag{15b}$$

and the integral conservation condition (12c) transformed into the boundary condition at $x=0$ of the form

$$Q|_{x=0}=1. \tag{15c}$$

Introducing the Lagrange variables (y,T) as

$$y=x\exp\left(-\int_0^\tau[\alpha(s)-1]ds\right)+\int_0^\tau\exp\left(-\int_0^t[\alpha(s)-1]ds\right)dt, \tag{16a}$$

$$T=\tau \tag{16b}$$

and the new unknown function $V(y,T)$ as

$$V=Q\exp\left(-3\int_0^T\alpha(s)ds\right), \tag{17}$$

we obtain the following free boundary problem for V and α:

$$V_T=0, \quad y_0(T)<y<\infty, \tag{18a}$$

$$V|_{y=y_0}=\exp\left(-3\int_0^T\alpha(s)ds\right), \tag{18b}$$

$$y_0(T)=\int_0^T\exp\left(-\int_0^t[\alpha(s)-1]ds\right)dt, \tag{18c}$$

$$V|_{T=0}=Q_0(y). \tag{18d}$$

Straightforward analysis of Eqs. (18a)–(18d) yields the following equation for α:

$$\exp\left(-3\int_0^T \alpha(s)ds\right)=Q_0\left[\int_0^T \exp\left(-\int_0^t[\alpha(s)-1]ds\right)dt\right]. \tag{19}$$

To solve this equation let us define $T_0(l)$ as the time at which the free boundary $y=y_0(T)$ reaches the point $y=l$. We have from Eqs. (18c), (18d), (19)

$$l=\int_0^{T_0(l)}\exp\left(-\int_0^t[\alpha(s)-1]ds\right)dt, \tag{20a}$$

$$Q_0(l)=\exp\left(-3\int_0^{T_0(l)}\alpha(s)ds\right). \tag{20b}$$

The solution of Eqs. (20a), (20b) yields

$$T_0(l)=\ln\left(1+\int_0^l Q_0^{-1/3}(s)ds\right), \tag{21a}$$

$$\alpha[T_0(l)]=-\frac{Q_0'(l)}{3Q_0^{2/3}(l)}\left(1+\int_0^l Q_0^{-1/3}(s)ds\right). \tag{21b}$$

An obvious consequence of Eqs. (21a), (21b) is the following ‘‘weak’’ selection principle: if

$$Q_0(x)=\begin{cases}O[(x_o-x)^\gamma] & \text{for } x\to x_0-0\\ 0 & \text{for } x>x_0\end{cases}$$

or

$$u_0(x)=\begin{cases}O[(x_o-x)^{\gamma-4}] & \text{for } x\to x_0-0\\ 0 & \text{for } x>x_0\end{cases} \tag{22}$$

then

$$\lim_{l\to x_0}T_0(l)=\infty \tag{23a}$$

and

$$\lim_{t\to\infty}\alpha(t)=\frac{\gamma}{\gamma-3}. \tag{23b}$$

We reiterate (see the Introduction) the ambiguity of this ‘‘weak’’ selection principle and of the related nonuniqueness of asymptotics, as dependent on such an essentially unmeasurable characteristic as the value of the logarithmic derivative of the initial profile at the support boundary. This ambiguity is to be removed by the ‘‘strong’’ selection principle (see Ref. [10]) analyzed below (Sec. III), whereas we conclude this section by analyzing the steady-state solutions of the diffusionally perturbed counterpart of the problem (15a)–(15c) of the form

$$Q_\tau-3\alpha(\tau)Q+[x(\alpha(\tau)-1)-1]Q_x=\varepsilon^2Q_{xx}, \quad 0<x<\infty, \tag{24a}$$

$$Q|_{\tau=0}=Q_0(x), \tag{24b}$$

$$Q|_{x=0}=1. \tag{24c}$$

The steady-state version of Eqs. (24a)–(24c) reads

$$\varepsilon^2Q_{xx}+3\alpha Q-[x(\alpha-1)-1]Q_x=0, \quad 0<x<\infty, \tag{25a}$$

$$Q(0)=1. \tag{25b}$$

The solution to this problem for $\alpha=1$ is

$$Q=C_1\exp(r_-x)+(1-C_1)\exp(r_+x) \tag{26}$$

with $r_\pm=(-1\pm\sqrt{1-12\varepsilon^2})/2\varepsilon^2$. Taking into account boundedness of $u=d^4Q/dx^4$ we find that $C_1=O(\varepsilon^8)$ and thus, to the leading order in ε,

$$\lim_{\varepsilon\to 0}Q(x)=\exp(-3x) \tag{27}$$

in accordance with the respective expression in Eq. (13). Solving Eqs. (25a), (25b) for $\alpha<1$ we find

$$Q=(-x\gamma+1)^{-\frac{1}{2}}\exp\left(-\frac{[-x\gamma+1]^2}{4\gamma\varepsilon^2}\right)\times Y\left(-\frac{3(\gamma+1)}{2\gamma}-\frac{1}{4},\frac{1}{4},-\frac{[-x\gamma+1]^2}{2\gamma\varepsilon^2}\right) \tag{28a}$$

with $\gamma=\alpha-1$ and

$$Y(a,b,z)=C_1M_{a,b}(z)+C_2(C_1)M_{a,-b}(z), \tag{28b}$$

where $M_{a,b}(z)$, $M_{a,-b}(z)$ are the respective Whittaker's functions (see Ref. [11]) and constant C_2 is related to C_1 by means of Eq. (25b). Using asymptotic properties of the Whittaker's functions (see Ref. [11]), we find

$$Q(x)x=O(x^{-3\alpha/(1-\alpha)})\text{as } x\to\infty \tag{29}$$

and

$$\lim_{\varepsilon\to 0}Q(x)=[1-(\alpha-1)x]^{-3\alpha/(1-\alpha)}. \tag{30}$$

The last case to be considered is $1<\alpha<4$, corresponding to the compact support solutions of the unperturbed steady-state problem with the respective power-law vanishing at the right edge of the support. In this case we find

$$Q=\begin{cases}(-x\gamma+1)^{-1/2}\exp\left(-\frac{[-x\gamma+1]^2}{4\gamma\varepsilon^2}\right)Y\left(-\frac{3(\gamma+1)}{2\gamma}-\frac{1}{4},\frac{1}{4},-\frac{[-x\gamma+1]^2}{2\gamma\varepsilon^2}\right) & \text{for } 0\leqslant x<\frac{1}{\gamma}=\frac{1}{\alpha-1}\\ 0 & \text{for } x\geqslant\frac{1}{\gamma},\end{cases} \quad (31)$$

with

$$Y(a,b,z)=C_1M_{a,b}(z). \quad (32)$$

Making once more use of the asymptotic properties of the Whittaker's functions, we obtain

$$Q(x)=C_1O\left[\exp\left(\frac{(1-x\gamma)^2}{4\gamma\varepsilon^2}\right)\right]$$

$$\text{as } \varepsilon\to 0 \text{ for every finite } x>\frac{1}{\gamma}. \quad (33)$$

We conclude that for boundedness in the limit $\varepsilon\to 0$ we must have $\lim_{\varepsilon\to 0}C_1=0$ which is incompatible with boundary condition (25b). Thus, no similarity solutions with a finite support, but the one with $\alpha=1$, may be recovered as the $\varepsilon\to 0$ limits of the respective steady state solutions in the diffusionally perturbed model. In other words, these solutions, corresponding to the critical size parameter α in the range $1<\alpha<4$ and characterized by a power law vanishing of the solution profile at the support boundary are unstable with respect to small diffusional perturbation.

III. STRONG SELECTION IN DIFFUSIONALLY PERTURBED PROBLEM

In this section we analyze the time-dependent diffusionally perturbed problem (11a)–(11b), with $N=-1$, in the form (24a)–(24c) and study the mechanism of ''strong'' selection by which the limiting value $\alpha=1$ of the critical size parameter is picked through time evolution. The existence of this ''strong'' selection is suggested by numerical solution of the time-dependent diffusionally perturbed problem and by the respective limiting results for steady-state solutions of the previous section.

In terms of the Lagrange variables (y,T), $V(y,T)$, defined by Eqs. (16a), (16b), (17), Eqs. (24a)–(24c) yield the following free boundary problem:

$$V_T=\varepsilon^2\exp\left(-2\int_0^T[\alpha(s)-1]ds\right)V_{yy},\quad 0<y_0(T)<y<\infty,$$

$$T>0, \quad (34a)$$

$$V|_{y=y_0}=\exp\left(-3\int_0^T\alpha(s)ds\right), \quad (34b)$$

$$y_0(T)=\int_0^T\exp\left(-\int_0^p[\alpha(s)-1]ds\right)dp, \quad (34c)$$

$$V|_{T=0}=Q_0(y). \quad (34d)$$

First, let us consider the diffusionally perturbed evolution of a similarity solution with a finite support and a power law vanishing at the support's right edge, corresponding to a nonlimiting value of α:

$$\alpha(0)=\alpha_0\in(1,4), \quad (35a)$$

$$V|_{T=0}=Q_0=\begin{cases}[1-y(\alpha_0-1)]^{3\alpha_0/(\alpha_0-1)} & \text{for } 0\leqslant y<\frac{1}{\alpha_0-1},\\ 0 & \text{for } y\geqslant\frac{1}{\alpha_0-1}.\end{cases} \quad (35b)$$

[In what follows we generalize our analysis to arbitrary initial data with a compact support and a diffusional perturbation in physical (nonsimilarity) variables.] The initial data (35b) is a regular smooth function for all values of y except the point $\tilde{y}\overset{\text{def}}{=}1/(\alpha_0-1)$, around which an order ε wide internal layer is formed.

The outer problem, valid for all (y,T) such that $\tilde{y}-y\gg\varepsilon$, is the unperturbed problem (18a)–(18d) and its solution is the respective similarity solution $[Q_0(y),\alpha_0]$. Hence,

$$V(y,T)=Q_0(y)\quad\text{for all } y\geqslant y_0 \text{ and } \tilde{y}-y\gg\varepsilon, \quad (36a)$$

$$\alpha(T)=\alpha_0 \text{ for all } T>0 \text{ such that } \tilde{y}-y_0(T)\gg\varepsilon, \quad (36b)$$

$$y_0(T)=\frac{\exp[-(\alpha_0-1)T]}{\alpha_0-1}$$

$$\text{for all } T>0 \text{ such that } \tilde{y}-y_0(T)\gg\varepsilon \quad (36c)$$

and, thus, α and V are constant in time for all values of $T<O(|\ln(\varepsilon)|)$.

Let us consider the inner problem whose solution affects the value of α for $T\geqslant O(|\ln(\varepsilon)|)$. In terms of the inner space and time variables

$$z=\frac{y-\tilde{y}}{\varepsilon}, \quad (37a)$$

$$\tilde{T}=\int_0^T\exp\left(-2\int_0^p[\alpha(s)-1]ds\right)dp \quad (37b)$$

this problem reads, to the leading order in ε,

$$W_{\tilde{T}}=W_{zz},-\infty<z<\infty, \quad (38a)$$

$$W|_{\tilde{T}=0} = \begin{cases} (\alpha_0 - 1)^{\beta} \varepsilon^{\beta} (-z)^{\beta} & z<0, \\ 0 & z>0, \end{cases} \tag{38b}$$

where $\beta = 3\alpha_0/(\alpha_0 - 1)$ and $W(z,\tilde{T}) = V(y,T)$. Integration of Eqs. (38a), (38b) yields

$$W(z,\tilde{T}) = \frac{(\alpha_0-1)^{\beta}}{2\sqrt{\pi\tilde{T}}}\varepsilon^{\beta}\int_{-\infty}^{0}(-\xi)^{\beta}\exp\left(-\frac{(z-\xi)^2}{4\tilde{T}}\right)d\xi. \tag{39}$$

Let us recall that for $T < O(|\ln(\varepsilon)|)$, the critical size parameter α is almost constant and, thus,

$$\tilde{T} = \frac{1}{2(\alpha_0-1)} 1 - \exp[-2(\alpha_0-1)T] = \frac{1}{\tilde{y}^2} - [y_0(T) - \tilde{y}]^2. \tag{40}$$

It follows from Eq. (40) that for T still the interval $O(1) < T < O(|\ln(\varepsilon)|)$, when the inner layer still has no effect on α, the inner time $\tilde{T}$ already approaches its limiting value $1/2(\alpha_0 - 1)$, that is,

$$\left|\tilde{T} - \frac{1}{\tilde{y}^2}\right| \ll 1. \tag{41}$$

Furthermore, taking into account Eq. (37b) we observe that inequality (41) holds for all T in the interval $O(1) < T < O(1/\varepsilon^2)$. This implies that by the time T when the free boundary $y_0(T)$ reaches the internal layer, the inner time $\tilde{T}$ is already at its limiting value and the solution of the inner problem ceases to evolve. Thus, we conclude that for all $T > O(1)$ the solution of the problem (34a)–(34c) is, to the leading order in ε, constant in time everywhere including the internal layer in which

$$V(y,T) \approx W\left(\frac{y-\tilde{y}}{\varepsilon}, \frac{1}{\tilde{y}^2}\right) \tag{42a}$$

with the free boundary y_0 and critical size parameter α determined by the equalities

$$\exp\left(-3\int_0^T a(s)ds\right) \approx W\left(\frac{y_0-\tilde{y}}{\varepsilon}, \frac{1}{\tilde{y}^2}\right), \tag{42b}$$

$$y_0(T) = \int_0^T \exp\left(-\int_0^p [\alpha(s)-1]ds\right)dp. \tag{42c}$$

Summarizing, the solution to the diffusionally perturbed problem (34a)–(34d) reads

$$V = \tilde{V} + O(\varepsilon), \tag{43a}$$

$$\alpha = \tilde{\alpha} + O(\varepsilon). \tag{43b}$$

Here $\tilde{V}$ and $\tilde{\alpha}$ are the solutions to the unperturbed problem (18a)–(18c), with the initial data regularized in the vicinity of the support boundary $\tilde{y}$ so that

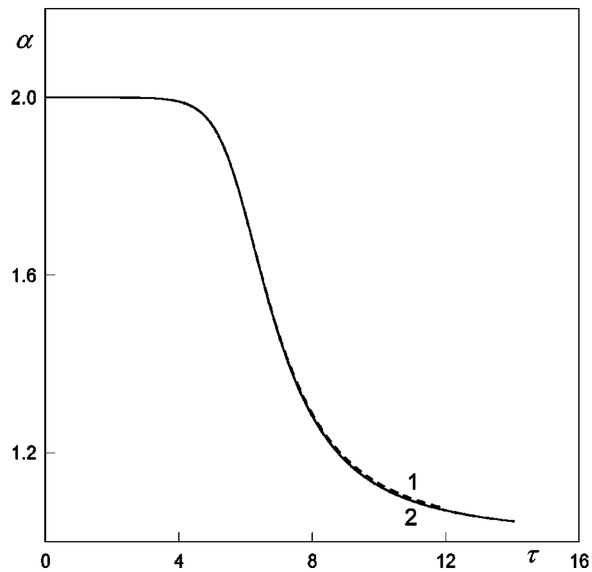

FIG. 1. Time evolution of the critical size parameter α in the approximate analytical solution (line 1) to the diffusionally perturbed problem (34a)–(34d) for $\alpha_0 = 2$ and $\varepsilon = 10^{-3}$, and in the respective exact numerical solution (line 2).

$$\tilde{V}(y,0) = Q_0(y) + W\left(\frac{y-\tilde{y}}{\varepsilon}, \frac{1}{\tilde{y}^2}\right) - W\left(\infty, \frac{1}{\tilde{y}^2}\right). \tag{44}$$

The respective leading order approximate solutions $\tilde{V}(y,T)$, $\tilde{\alpha}(T)$ are given by Eqs. (21a)–(21b). For comparison, we present in Fig. 1 the numerical solution of full diffusionally perturbed problem (34a)–(34d) for $\alpha_0 = 2$ and $\varepsilon = 10^{-3}$ along with a respective approximate solution.

In the rest of this section we consider the diffusional perturbation of the OR model with a general initial distribution on a compact support and a power law vanishing of the distribution profile at the support boundary for any integer value of the parameter N in the range $N = -1,0,1,2$. We begin with consideration of the case $N = -1$ with the initial data of the form

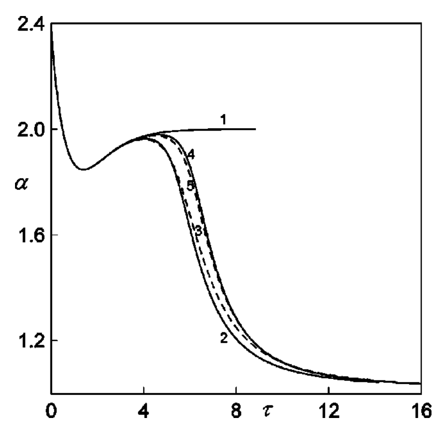

FIG. 2. Time evolution of the critical size parameter α in the unperturbed problem (line 1), in the exact numerical solution to the diffusionally perturbed problem for $\varepsilon = 10^{-3}$ (line 2) and $\varepsilon = 10^{-4}$ (line 4), and in the respective approximate analytical solutions for $\varepsilon = 10^{-3}$ (line 3) and $\varepsilon = 10^{-4}$ (line 5).

$$V|_{T=0}=V_0(y)=\begin{cases} O[(\tilde{y}-y)^\gamma], & \gamma>4 \quad \text{for } y\to\tilde{y}-0,\\ 0 & \text{for } y>\tilde{y}. \end{cases} \tag{45}$$

We distinguish the following three time scales.

(1) *Initial phase* $T=O(1)$. $\alpha(T)$ defined by the solution of the outer (unperturbed) problem. There is no effect of diffusional perturbation for T in this range.

(2) *Transient state*: $O(1)<T<O(|\ln\varepsilon|)$. Still, no effect of diffusional perturbation upon $\alpha(T)$. $\alpha(T)$ reaches its transient asymptotic value

$$\alpha(T)\to\alpha_0=\frac{\gamma}{\gamma-3}. \tag{46}$$

(3) *Final asymptotic state*. $T\geqslant O(|\ln\varepsilon|)$. $\alpha(T)$ tends to its limiting value 1, following the scenario described above.

Summarizing, in the case of a general initial distribution function $V_0(y)$, satisfying condition (45), representation (43a), (43b) of the solution to the full diffusionally perturbed problem still holds, with the following modification of the regularized initial data:

$$\tilde{V}|_{T=0}=\begin{cases} V_0-(\tilde{y}-y)^\gamma \lim\limits_{y\to\tilde{y}-}\dfrac{V_0(y)}{(\tilde{y}-y)^\gamma}+W\left(\dfrac{y-\tilde{y}}{\varepsilon},T_0\right)\lim\limits_{y\to\tilde{y}-}\dfrac{V_0(y)}{(\tilde{y}-y)^\gamma}/(\alpha_0-1)^\gamma & \text{for } y<\tilde{y},\\ W\left(\dfrac{y-\tilde{y}}{\varepsilon},T_0\right)\lim\limits_{y\to\tilde{y}-}\dfrac{V_0(y)}{(\tilde{y}-y)^\gamma}/(\alpha_0-1)^\gamma & \text{for } y>\tilde{y}. \end{cases} \tag{47a}$$

Here

$$T_0=\int_0^\infty \exp\left(-2\int_0^t[\tilde{\alpha}(s)-1]ds\right)dt \tag{47b}$$

and $\tilde{\alpha}(T)$ is the critical size parameter obtained by the solution of the respective unperturbed problem

$$\lim_{T\to\infty}\tilde{\alpha}(T)=\alpha_0. \tag{47c}$$

In the Fig. 2 we present a comparison of a numerical solution to the diffusionally perturbed problem (34a)–(34d) with the respective approximate solution (43a), (43b), (47a)–(47c) for the initial distribution

$$V_0(y)=\begin{cases}(1-y)^2+6(1-y)^4 & \text{for } 0\leqslant y<1,\\ 0 & \text{for } 1\leqslant y, \quad \varepsilon=10^{-3} \text{ and } \varepsilon=10^{-4}.\end{cases}$$

The obtained mechanism of ‘‘strong’’ selection of the limiting value of the critical size parameter $\alpha=1$ is sufficiently general and, in particular, also holds for different types of diffusional perturbation. Thus, let us consider a general time dependent diffusivity of the type

$$D=\varepsilon^2 k(T). \tag{48}$$

Repeating the above arguments with slight modifications, we find that the representation (47a) for the initial data of the approximate solution is still valid with only the value of ‘‘stabilization’’ time T_0 modified in accordance with Eq. (48). For example, we have

$$T_0=\int_0^\infty \exp(-t)\exp\left(-2\int_0^t[\tilde{\alpha}(s)-1]ds\right)dt \tag{49}$$

for $k(T)=\exp(-T)=1/t$ corresponding to a constant diffusivity in the original physical variables (x,t).

There is no analytical representation for the solution of the unperturbed OR model problem with $N>-1$ (Wagner model $N=0$, LS model $N=1$) and we cannot find the exact value of the ‘‘stabilization’’ time T_0 in this case. However, universal character of the described mechanism of ‘‘strong selection’’ for different N is supported by comparison of numerical solution in the diffusionally perturbed models of OR with the respective numerically obtained approximate solutions.

Thus, let us consider, for example, the LS model of OR $(N=1)$. Introducing a new unknown

$$Q=\int_x^\infty u\,dx \tag{50}$$

and integrating the counterpart of Eq. (11a) with $N=1$, we obtain the following version of the diffusionally perturbed problem (11a), (11b):

$$Q_\tau-Q+\left(\frac{1}{x}\left[\alpha(\tau)-\frac{1}{x}\right]-\frac{x}{3}\right)Q_x=\varepsilon^2 k(\tau)Q_{xx},$$

$$0<x<\infty, \quad \tau>0, \tag{51a}$$

$$\int_0^\infty Qx^2 dx = 2, \tag{51b}$$

$$Q|_{\tau=0} = Q_0(x) = \begin{cases} O[(\tilde{x}-x)^\gamma], & x\to\tilde{x}-0, \\ 0, & x>\tilde{x}. \end{cases} \tag{51c}$$

[Nonlinearity of Eq. (51a) in x does not allow us its further integration analogous to Eq. (14).] Defining $V^0(x,\tau)$, $\alpha^0(\tau)$ as the solution of the respective unperturbed problem, we find the Lagrange variables (y,T): $y=y(x,\tau)$, $T=\tau$ as a solution to the equation

$$y_\tau + \left[\frac{1}{x}\left(\alpha^0(\tau)-\frac{1}{x}\right)-\frac{x}{3}\right]y_x = 0 \tag{52a}$$

with initial condition

$$y|_{\tau=0} = x. \tag{52b}$$

In terms of these variables, problem (51a)–(51c) assumes the form

$$V_T = \begin{cases} 0, & y_0(T)<y, \quad \tilde{y}-y<O(\varepsilon) \\ \varepsilon^2 k(T) y_x^2(\tilde{y},T) V_{yy}, & \tilde{y}-y \geqslant O(\varepsilon), \quad T>0, \end{cases} \tag{53}$$

$$\int_{y_0(T)}^\infty V\frac{x(y,T)^2}{y_x} dy = \exp(T), \tag{54a}$$

$$V|_{T=0} = V_0(y) = \begin{cases} O[(\tilde{y}-y)^\gamma], & 0<y\leqslant\tilde{y}, \\ 0, & y>\tilde{y}, \end{cases} \tag{54b}$$

Here $V(y,T)=\exp(-T)Q(x,\tau)$. The stabilization time T_0 is

$$T_0 = \int_0^\infty k(T) y_x^2(\tilde{y},T)dT \tag{55}$$

and the approximate solution is found as that to the respective unperturbed problem with the following regularized initial data:

$$\tilde{V} = \begin{cases} V_0 - (\tilde{y}-y) \lim\limits_{y\to\tilde{y}-} \dfrac{V_0(y)}{(\tilde{y}-y)^\gamma} + W\left(\dfrac{y-\tilde{y}}{\varepsilon}, T_0\right), & 0<y\leqslant\tilde{y}, \\ W\left(\dfrac{y-\tilde{y}}{\varepsilon}, T_0\right), & y\geqslant\tilde{y}. \end{cases} \tag{56}$$

Here W is the solution of the inner problem

$$W_{\tilde{T}} = W_{zz}, \quad -\infty<z<\infty, \quad \tilde{T}>0, \tag{57a}$$

$$W|_{\tilde{T}=0} = \begin{cases} \lim\limits_{y\to\tilde{y}-} \dfrac{V_0(y)}{(\tilde{y}-y)^\gamma}(-z)^\gamma\varepsilon^\gamma, & z<0, \\ 0, & z>0. \end{cases} \tag{57b}$$

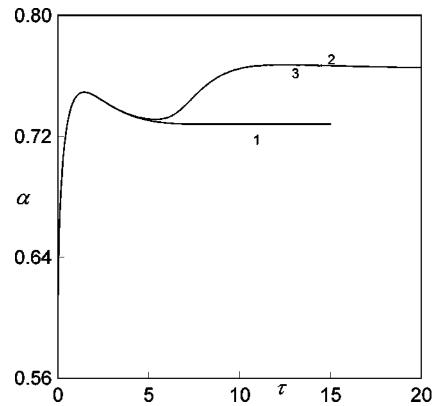

FIG. 3. Time evolution of the critical size parameter α in the unperturbed LS problem (line 1), in the numerical solution of the diffusionally perturbed problem for $\varepsilon=10^{-3}$ (line 2), and in the respective approximate solution (line 3).

z and $\tilde{T}$ are, respectively, the inner space and time variables, defined as

$$z = \frac{y-\tilde{y}}{\varepsilon}, \tag{58a}$$

$$\tilde{T} = \int_0^T k(s) y_x^2(\tilde{y},s) ds. \tag{58b}$$

In particular, for a diffusional perturbation in similarity variables $(k=1)$ we have

$$\tilde{T} = \frac{1}{2(\alpha/\alpha_2^2-1)}\{1-\exp[2(\alpha/\alpha_2^2-1)T]\} \tag{59a}$$

with

$$a_2 = \tilde{x} < \sqrt{\alpha} \quad \text{for} \quad \alpha > \alpha_{\lim} = \left(\frac{3}{2}\right)^{2/3} \tag{59b}$$

[see Eqs. (7)–(9)].

In Fig. 3 we compare the numerical solution to the time-dependent diffusionally perturbed LS model $(N=1,\ \varepsilon=10^{-3})$ with the respective approximate solution, namely, that of the unperturbed problem with regularized initial data, Eq. (56), and $u_0(x)$ of the form

$$u_0(x) = \begin{cases} (1-x)^2, & 0\leqslant x\leqslant 1, \\ 0, & x\geqslant 1. \end{cases} \tag{60}$$

IV. CONCLUSIONS

(1) As conjectured previously by Meerson [10], fluctuations, accounted for as a diffusional perturbation in the LSW model of Ostwald ripening, yields a strong selection of the limiting LSW similarity solution as a sole attractor in the time-dependent problem; on the other hand, the weak selection of other similarity solutions comes up in this process as intermediate time asymptotics for the initial conditions with a compact support.

(2) The universal (independent of the perturbation parameter ε) characteristic of the strong selection process is the scaled internal layer solution profile and, in particular, its intrinsic inner time asymptotics (42a). Namely this latter governs, in accordance with Eq. (42b), the transition of the critical size parameter α from its intermediate asymptotic value, picked by the weak selection, to the final limiting LS–W value.

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